

# Continuous-Time Consensus under Non-Instantaneous Reciprocity

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**Abstract**—We consider continuous-time consensus systems whose interactions satisfy a form of reciprocity that is not instantaneous, but happens over time. We show that these systems have certain desirable properties: They always converge independently of the specific interactions taking place and there exist simple conditions on the interactions for two agents to converge to the same value. This was until now only known for systems with instantaneous reciprocity. These results are of particular relevance when analyzing systems where interactions are a priori unknown, being for example endogenously determined or random. We apply our results to an instance of such systems.

## I. INTRODUCTION

We consider systems where  $n$  agents each have a value  $x_i \in \mathbb{R}$  that evolves according to

$$\dot{x}_i = \sum_{j=1}^n a_{ij}(t)(x_j(t) - x_i(t)), \quad (1)$$

where the  $a_{ij}(t) \geq 0$  are non-negative functions of time. This means that the value of  $x_i$  is continuously attracted by the values of the agents  $j$  for which  $a_{ij}(t) \neq 0$ . These systems are called consensus systems because the interactions tend to reduce the disagreement between the interacting agents, and because any consensus state where all  $x_i$  are equal is an equilibrium of the system. Analogous systems also exist in discrete time [14], [20], [27]. Consensus systems play a major role in decentralized control [16], data fusion [3], [30] and distributed optimization [9], [21], but also when modeling some animal [7], [28] or social phenomena [5], [17].

General convergence results for consensus systems involve connectivity assumptions that are hard to check for state-dependent interactions, and do not allow treating clustering phenomena. As detailed in the state of the art, more recent results guarantee convergence to one or several clusters under various assumptions on the symmetry or reciprocity of the interactions. All these reciprocity properties have however to be satisfied instantaneously and at every time. We extend

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them to treat systems where reciprocity is not instantaneous but happens on average over time.

This extension only holds under certain assumptions on the way reciprocity occurs. Indeed, non-instantaneous reciprocity may fail to ensure convergence and lead to oscillatory behaviors when the interaction weights are not properly bounded, or when the time periods across which it occurs grow unbounded (see Section III-B for an example). To prove our result we show that, for an appropriate sequence of times  $t_k$ , the states  $x(t_k)$  can be seen as the trajectory of a certain discrete time consensus system. By analyzing the effect of each matrix of this system on some artificial initial conditions, we obtain bounds on their coefficients, and show that this system satisfies reciprocity conditions guaranteeing convergence.

The rest of the paper is organized as follows. The introduction includes a state of the art on consensus systems, a section pointing out the interest of non-instantaneous reciprocity and a summary of our contributions. Section II formally introduces the system that we are considering and presents our main results. Examples illustrating our results and the necessity of an underlying assumption are then presented in Section III. In Section IV, we demonstrate the use of our results on a specific multi-agent applications. Sections V and VI contain the proofs, and we finish by some conclusions in Section VII.

### State of the art

Consensus systems have been the object of many studies during the recent years, focusing particularly on finding conditions under which the system converges, possibly to a consensus state, but also on the speed of convergence. Classical results typically guarantee convergence to consensus under some (repeated) connectivity conditions on the interactions, see for example [14], [20], [29] or [22], [23] for surveys.

Different recent works have shown that stronger results hold when the interactions satisfy some form of reciprocity. Hendrickx and Tsitsiklis have for example introduced the *cut-balance* assumption on the interactions [13], stating that there exists a  $K$  such that for every subset  $S$  of agents and time  $t$ , there holds

$$\sum_{i \in S, j \notin S} a_{ij}(t) \leq K \sum_{i \in S, j \notin S} a_{ji}(t). \quad (2)$$

This assumption can actually be shown to mean that whenever an agent  $i$  influences agent  $j$  indirectly, agent  $j$  also influences agent  $i$  indirectly, with an intensity that is within a

constant ratio of that of  $i$  on  $j$ . Particular case of this assumptions include symmetric interactions  $a_{ij} = a_{ji}$ , bounded-ratio symmetry  $a_{ij} \leq K a_{ji}$ , or any average-preserving dynamics  $\sum_j a_{ij} = \sum_j a_{ji}$  for every  $i$ . It was shown in [13] that systems satisfying the cut-balance assumption (2) always converge, though not necessarily to consensus. Moreover, two agents' values converge to the same limiting value if they are connected by a path in the graph of *persistent* interactions (also called *unbounded* interactions in the literature), defined by connecting  $i$  and  $j$  if  $\int_{t=0}^{\infty} a_{ij}(t)$  is infinite. These results allow analyzing the convergence properties of systems with relatively complex interactions; see the discussion in [13] for an example in opinion dynamics, or [8] for an application to system involving event-based ternary control of second order agents.

Martin and Girard have later shown [18] that in the case of convergence to a global consensus, the cut-balance assumption could be weakened, allowing for the interaction ratio bound  $K$  to slowly grow with the amount of interactions that have already taken place in the system. They also provide an estimate of the convergence speed in terms of the interactions having taken place.

Related convergence results were also proved for systems involving a continuum of agents under a strict symmetry assumption in [11]. Finally, we note that similar results of convergence under some reciprocity conditions have been obtained for discrete time consensus systems, see for example [2], [15], [20], [25], [26]. However, none of these results allow for non-instantaneous reciprocity.

### Non-instantaneous reciprocity

All the results taking advantage of reciprocity require the reciprocity condition to be satisfied instantaneously at (almost) all times. They would thus not apply to systems that are essentially reciprocal, but where the reciprocity may be delayed, or happen over time. In systems relying on certain wired or wireless network protocols, agents may be unable to simultaneously send and receive information, resulting in loss of instantaneous reciprocity, even if the interactions are meant to be reciprocal. Non-instantaneous reciprocity also arises in a priori symmetric systems where the control of the agents is event-triggered or self-triggered. Indeed, suppose that at some time the conditions are such that agents  $i$  and  $j$  should interact. It is very likely that one agent will update its control action before the other, so that during a certain interval of time the actual interactions will not be symmetric.

Similar problems are present in systems prone to occasional failures, or unreliable communications, where the communication between two agents can temporarily be interrupted in one direction for a limited amount of time.

Issues with non-instantaneous reciprocity may also arise in swarming processes or in any multi-agent control problem due to the limited scope of sensors. Suppose indeed that the sensors are not omnidirectional, as it is for example the case for human or animal eyes. It is then generally impossible for an agent to observe all its neighbors at the same time. The same issue arises if the agent can only treat a limited number

of neighbors simultaneously. A natural solution is then to observe a subset of the neighbors and to periodically modify the subset being observed. This can for example be achieved by continuously rotating the directions in which observations are made. In that case, even if the neighborhood relation is symmetrical, it is again highly likely that an agent  $i$  will sometime observe an agent  $j$  without that  $j$  is observing  $i$  at that particular moment, but that  $j$  will observe  $i$  later. In all these situations, one could hope to take advantages of the essential reciprocity of the system design even if this reciprocity is not always instantaneously satisfied.

### Contributions

We show in our main result (Theorem 1) that the convergence of systems of the form (1) is still guaranteed if the system satisfies some form of non-instantaneous reciprocity, or reciprocity on average. More specifically, we assume that the cut-balance condition (2) is satisfied *on average* on a sequence of contiguous intervals. These intervals can have arbitrary length, but the amount of interaction taking place during each of them should be uniformly bounded. Under these assumptions, we show that the system always converges. Moreover, two agent values converge to the same limit if they are connected by a path in the graph of persistent interactions, defined by connecting two agents  $i, j$  if  $\int_{t=0}^{\infty} a_{ij}(t)dt$  is infinite.

We also particularize our general result to systems satisfying a form of pairwise reciprocity over bounded time intervals. This particularized result is more conservative, but often easier to check. We illustrate it on an application.

## II. PROBLEM STATEMENT AND MAIN RESULT

We study the integral version of the consensus system (1):

$$x_i(t) = x_i(0) + \int_0^t \sum_{j=1}^n a_{ij}(s)(x_j(s) - x_i(s))ds, \quad (3)$$

where for all  $i, j \in \mathcal{N} = \{1, \dots, n\}$ , the *interaction weight*  $a_{ij}$  is a non-negative measurable function of time, summable on bounded intervals of  $\mathbb{R}^+$ . There exists a unique function of time  $x : \mathbb{R}^+ \rightarrow \mathbb{R}^n$  which satisfies for all  $t \in \mathbb{R}^+$  the integral equation (3), and it is locally absolutely continuous (see Theorem 54 and Proposition C.3.8 in [24, pages 473-482]). This function is actually the Caratheodory solution to the differential equation (1) and can equivalently be defined as absolutely continuous function satisfying (1) at almost all times. We call it the *trajectory* of the system.

Following the discussion in section I, we introduce a new condition that allows for non-instantaneous reciprocity of interactions, i.e., we only require that the reciprocity occurs on the integral weights  $\int a_{ij}(s)ds$  over some bounded time intervals.

*Assumption 1 (Integral weight reciprocity):* There exists a sequence  $(t_p)_{p \in \mathbb{N}}$  of increasing times with  $\lim_{p \rightarrow +\infty} t_p = +\infty$  and some uniform bound  $K \geq 1$  such that, for all non-

empty proper subsets  $S$  of  $\mathcal{N}$ , and for all  $p \in \mathbb{N}$ , there holds

$$\sum_{i \in S, j \notin S} \int_{t_p}^{t_{p+1}} a_{ij}(t) dt \leq K \sum_{i \in S, j \notin S} \int_{t_p}^{t_{p+1}} a_{ji}(t) dt. \quad (4)$$

Assumption 1 generalizes most types of reciprocity found in the consensus literature. In particular, it generalizes the cut-balance assumption (2) developed in [13] and discussed in the Introduction.

We will see in a simple example in section III-B that Assumption 1 alone is not sufficient to guarantee the convergence of the system. We need to further assume that the integral of the interactions taking place in each interval  $[t_p, t_{p+1}]$  is uniformly bounded.

*Assumption 2 (Uniform upper bound on integral weights):* The sequence  $(t_p)$  used in Assumption 1 is such that

$$\int_{t_p}^{t_{p+1}} a_{ij}(t) dt \leq M,$$

holds for all  $i, j \in \mathcal{N}$ ,  $p \in \mathbb{N}$  and some constant  $M$ .

We now state our main result, whose proof is sketched in Section V.

*Theorem 1:* Suppose that the interaction weights of system (3) satisfy Assumptions 1 (integral reciprocity) and 2 (upper bound on weight integral). Then, every trajectory  $x$  of system (3) converges.

Moreover, let  $G = (\mathcal{N}, E)$  be the graph of persistent weights defined by connecting  $(j, i)$  if  $\int_0^\infty a_{ij}(t) dt = +\infty$ . Then, there is a directed path from  $i$  to  $j$  in  $G$  if and only if there is a directed path from  $j$  to  $i$ , and there holds in that case  $\lim_{t \rightarrow \infty} x_i(t) = \lim_{t \rightarrow \infty} x_j(t)$ .

The second part of the theorem implies that there is a local consensus in each connected component<sup>1</sup> of the graph  $G$  of persistent interactions.

Checking that an actual decentralized system satisfies Assumptions 1 and 2 may sometimes be nontrivial, as these assumptions involve global conditions on specific time periods. When the interactions between agents are decentralized and depend on local parameters, the period over which the interactions between one pair of agents can be deemed reciprocal does not necessarily correspond the period over which the interactions between another pair of agents is reciprocal. One may then find it uneasy to select appropriate times  $t_p$  guaranteeing the reciprocity conditions for all pairs of agents simultaneously on the same intervals  $[t_p, t_{p+1}]$ .

We therefore introduce a new local assumption that we will show to be more conservative than Assumptions 1 and 2 when interactions are bounded. It requires that whenever an agent  $j$  influences an agent  $i$  at some time  $t$ , both agents should influence each other with a sufficient strength across a certain time interval around  $t$ .

<sup>1</sup>A connected component is a subgraph in which any two nodes are connected to each other by at least a path, and which is connected to no other nodes in the graph.

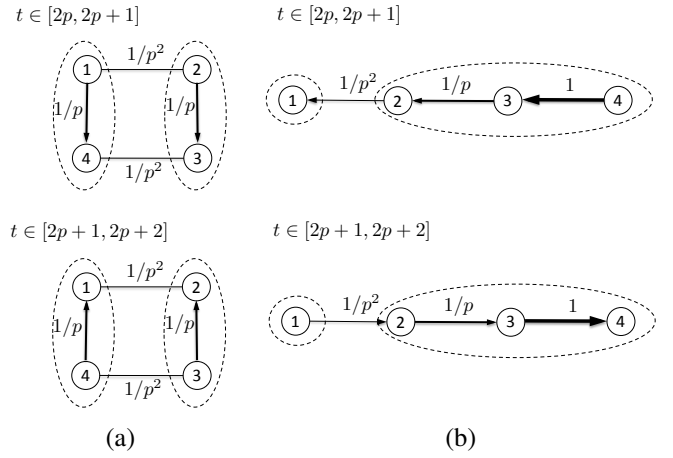


Fig. 1. Representations of the interactions taking place in example 1 (a) and in example 2 (b) in Section III-A, and of the connected components of the graph of persistent interactions, in which local consensuses occur.

*Assumption 3 (Pairwise reciprocity):* There exist constant  $\varepsilon, T > 0$  such that for every  $i, j \in \mathcal{N}$  and  $t$ , if  $a_{ij}(t) > 0$ , then there exists  $\underline{t}_{ij}, \bar{t}_{ij}$  such that

- $\bar{t}_{ij} - \underline{t}_{ij} \leq T$ ,
- $t \in [\underline{t}_{ij}, \bar{t}_{ij}]$ ,
- $\int_{\underline{t}_{ij}}^{\bar{t}_{ij}} a_{ij}(t) dt \geq \varepsilon$  and  $\int_{\underline{t}_{ij}}^{\bar{t}_{ij}} a_{ji}(t) dt \geq \varepsilon$ .

*Theorem 2:* Suppose that the interaction weights  $a_{ij}(t)$  of system (3) satisfy Assumption 3 and are uniformly bounded by some constant  $M'$ . Then they satisfy Assumptions 1 and 2, and the conclusions of Theorem 1 hold.

*Remark 1:* Theorem 1 and Theorem 2 are stated for systems where the coefficients  $a_{ij}(t)$  only depend on time, and the proof of Theorem 1 actually uses that fact. However, these results can directly be extended to solutions of systems with state-dependent coefficients  $\tilde{a}_{ij}(t, x)$ . Indeed, suppose that  $x$  is a solution of

$$x_i(t) = x_i(0) + \int_0^t \tilde{a}_{ij}(s, x(s))(x_j(s) - x_i(s)) ds, \quad (5)$$

then  $x$  is also a solution the linear time-varying systems (3) with ad hoc coefficients  $a_{ij}(t) = \tilde{a}_{ij}(t, x(t))$ , to which Theorem 1 applies. Similar extensions apply to randomized weights  $a_{ij}$ . Note however that the existence or uniqueness of a solution to nonlinear systems of the form (5) is in general a complex issue.

Finally, one can easily verify that Theorem 1 and Theorem 2 can be extended when the agent values  $x_i$  are in  $\mathbb{R}^n$  provided that the weights  $a_{ij}$  remain scalar. It suffices indeed in that case to apply the result to each component of the  $x_i$ .

### III. EXAMPLES

#### A. System with non-instantaneous reciprocity

In this Section, we present two simple 4-agents systems whose convergence can be established by Theorem 1 and by no other result on consensus available in the literature.

*Example 1:*

Our first example is depicted in Fig. 1(a). It contains two weakly interacting subsystems, inside each of which two agents successively attract each other. More specifically, the interactions start at time  $t = 2$  and are defined as follows: For every  $p \geq 1$ ,

- if  $t \in [2p, 2p + 2]$ ,  $a_{12} = a_{21} = a_{34} = a_{43} = 1/p^2$ ,
- if  $t \in [2p, 2p + 1]$ ,  $a_{32} = a_{41} = 1/p$ ,
- if  $t \in [2p + 1, 2p + 2]$ ,  $a_{23} = a_{14} = 1/p$ ,

and all values of  $a_{ij}(t)$  that are not explicitly defined are equal to 0. One can verify that this system satisfies Assumptions 1 and 2 with  $t_p = 2p$ ,  $K = 1$  and  $M = 2$ . We can thus apply Theorem 1 to establish its convergence. The graph of persistent interactions can also easily be built and contains the edges  $(2, 3)$ ,  $(3, 2)$ ,  $(1, 4)$  and  $(4, 1)$ . There are thus two connected components  $\{2, 3\}$  and  $\{1, 4\}$ , and two local consensuses  $x_2^* = x_3^*$  and  $x_1^* = x_4^*$ .

On the other hand, notice that the system does not satisfy any instantaneous reciprocity condition, so none of available reciprocity-based results applies. Moreau's result does not apply either due to the weak interactions in  $1/p^2$  between the subsystems (the interactions are not lower bounded ; see section 3.3 in [18] for a detailed explanation). Observe also that our result also applies if the interactions are interrupted during arbitrarily long periods. Suppose indeed that the interactions defined above do not take place during the intervals  $[2p, 2p + 1]$  and  $[2p + 1, 2p + 2]$  but during the intervals  $[p^2, p^2 + 1]$  and  $[p^2 + p, p^2 + p + 1]$ . Assumptions 1 and 2 still apply with  $t_p = p^2$ .

*Example 2:*

The second example involves a chain of four agents, which are attracted by their higher index neighbor for  $t \in [2p, 2p + 1]$  and their lower index neighbor for  $t \in [2p + 1, 2p + 2]$ , as depicted in Figure 1(b). Moreover, the ratios between weights of the different interactions grow unbounded.

Specifically, the interactions start again at  $t = 2$ , and for each  $p \geq 1$ ,

- if  $t \in [2p, 2p + 1]$ ,  $a_{12} = 1/p^2$ ,  $a_{23} = 1/p$  and  $a_{34} = 1$
- if  $t \in [2p + 1, 2p + 2]$ ,  $a_{21} = 1/p^2$ ,  $a_{32} = 1/p$  and  $a_{43} = 1$

and all values of  $a_{ij}(t)$  that are not explicitly defined are equal to 0. One can verify again that Assumptions 1 and 2 with  $t_p = 2p$ ,  $K = 1$  and  $M = 2$ , so that the convergence of the system follows from Theorem 1. The graph of persistent interactions contains the edges  $(2, 3)$ ,  $(3, 2)$ ,  $(3, 4)$  and  $(4, 3)$ , resulting in a local (trivial) consensus of agent 1, and a consensus between agent 2, 3 and 4.

Again, the system satisfies no instantaneous reciprocity condition, so none of available reciprocity-based results applies. Moreover, all the results of which we are aware and that do not rely on reciprocity require the interaction to be bounded from above and from below (see [19] for example). Since the ratios between the values of  $a_{34}$ ,  $a_{43}$  and  $a_{32}$ ,  $a_{23}$

grow unbounded, it would thus be impossible to apply them even to the connected component  $\{2, 3, 4\}$ , including if we re-scale the values of the coefficients by scaling time.

Besides, Theorem 1 would again apply exactly in the same way if the interactions were interrupted during arbitrary long periods of time

*B. Oscillatory behavior under integral reciprocity - Necessity of Assumption 2.*

The following Proposition formalizes the fact that Assumption 1 alone is not sufficient to guarantee convergence.

*Proposition 3:* There exist systems of the form (3) satisfying Assumption 1 (integral reciprocity) and that admit trajectories that do not converge.

To prove the Proposition, we present a 3-agent system which satisfies Assumption 1 (reciprocity) but whose trajectory does not converge. The idea is to have one agent (2) oscillating between two agents (1 and 3) that successively attract the former while remaining at a certain distance from each other, as depicted in Fig. 2. Agent 1 starts influencing 2. Since we only impose integral reciprocity,  $a_{12}$  and  $a_{21}$  do not have to be non-zero simultaneously. Also, because there is no uniform bound on influence, the distance between 2 and 1 has become arbitrarily close to 0 when agent 2 starts influencing back. So the overall influence of agent 2 over 1, this is  $\int a_{12} \cdot (x_2 - x_1)$ , is also arbitrarily small. This leads to an actual influence of 1 over 2 but not of 2 over 1. The same happens between 3 and 1, leading to convergence of 1 and 3 to distinct limits and oscillations of 2. We now present the formal proof.

*Proof:* Let  $(\rho_p)_{p \in \mathbb{N}}$  be a non-decreasing sequence such that  $\rho_p \geq 1$ , for all  $p \in \mathbb{N}$ . Let us consider a multi-agent system with 3 agents where  $x_1(0) = 0$ ,  $x_2(0) = 1/2$  and  $x_3(0) = 1$  and with the dynamics given by system (3) with weights

$$\begin{cases} \text{if } t \in [4p, 4p + 1), & a_{21}(t) = \rho_p, \\ \text{if } t \in [4p + 1, 4p + 2), & a_{12}(t) = \rho_p, \\ \text{if } t \in [4p + 2, 4p + 3), & a_{23}(t) = \rho_p, \\ \text{if } t \in [4p + 3, 4p + 4), & a_{32}(t) = \rho_p, \end{cases}$$

where only the non-zero weights have been detailed. Figure 2 illustrates the dynamics of this system.

Here, Assumption 1 holds with  $K = 1$  for  $t_p = 4p$ . It is easy to see that  $x_1(t)$  is non-decreasing,  $x_3(t)$  is non-increasing and for all  $t \geq 0$ ,  $x_1(t) \leq x_2(t) \leq x_3(t)$ . Integrating the dynamics of the system, we can show that for all  $p \in \mathbb{N}$ :

$$\begin{aligned} x_1(4p + 4) &= x_1(4p + 2) \leq x_2(4p + 2) = x_2(4p + 1) \\ &= (1 - e^{-\rho_p})x_1(4p) + e^{-\rho_p}x_2(4p) \\ &\leq (1 - e^{-\rho_p})x_1(4p) + e^{-\rho_p}x_3(0), \end{aligned}$$

and that

$$\begin{aligned} x_3(4p + 4) &\geq x_2(4p + 4) = x_2(4p + 3) \\ &= e^{-\rho_p}x_2(4p + 2) + (1 - e^{-\rho_p})x_3(4p + 2) \\ &\geq e^{-\rho_p}x_1(4p) + (1 - e^{-\rho_p})x_3(4p) \\ &\geq e^{-\rho_p}x_1(0) + (1 - e^{-\rho_p})x_3(4p). \end{aligned}$$

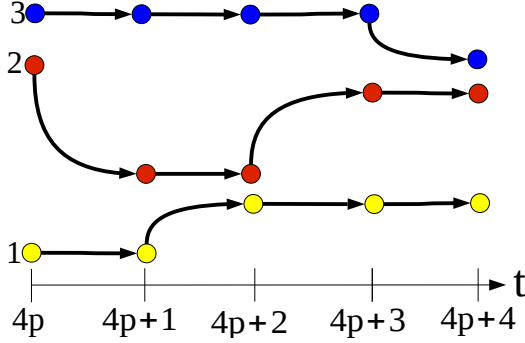


Fig. 2. Dynamics of the 3-agent system.

Combining the two previous results and initial conditions gives us

$$1 + (x_3(4p+4) - x_1(4p+4)) \geq (1 - e^{-\rho_p}) (1 + (x_3(4p) - x_1(4p))).$$

We observe that term  $1 + (x_3(4p) - x_1(4p))$  decreases more slowly than a geometric sequence of scale factor  $(1 - e^{-\rho_p})$ . Taking a sequence  $\rho_p$  growing sufficiently fast (and thus breaking the uniform bound Assumption 2) leads to convergence of this term arbitrarily close its initial value 2. Then,  $(x_3(4p))$  and  $(x_1(4p))$  do not converge to the same value. As a consequence, one can verify that  $x_2$  will keep oscillating between  $x_1$  and  $x_3$ . Hence, the system does not converge. ■

#### IV. APPLICATION TO MOBILE ROBOTS WITH INTERMITTENT ULTRASONIC COMMUNICATION

In this section we apply our results to a realistic system of mobile robots evolving in the plane  $\mathbb{R}^2$  and communicating using ultrasonic sensors. These sensors make for an affordable and thus widespread contactless mean of measuring distances [4], but are subject to certain limitation as detailed below. The objective of the group of robots is to achieve practical rendezvous, *i.e.* all robots should eventually lie in a ball of a certain maximal radius (see *e.g.* [6]). The robots have several functional constraints. The ultrasonic sensors in use are not accurate when measuring distances smaller than a radius  $d_0 > 0$ , thus we assume that the robots cannot make use of such measures and are blind at short range. Also, the robots' engines are limited and the velocity of each robots cannot exceed a maximum of  $\mu > 0$  in norm. Most importantly, in order to save energy, the robots activate their sensors intermittently, and in an asynchronous way: Robot  $i$  wakes up at every time  $t_k^i$ , and monitors its environment over the time-interval  $[t_k^i, t_k^i + \delta_{\min}]$ , for some  $\delta_{\min} > 0$ . (For simplicity, we take the same  $\delta_{\min}$  for every robot, but this is not crucial for our result). In addition, we assume that the sequence  $(t_k^i)$  satisfies  $t_{k+1}^i - t_k^i \in [\delta_{\min}, \delta_{\max}]$  for every  $k \in \mathbb{N}$ , for some  $\delta_{\max} > \delta_{\min}$ , and  $t_0^i \leq \delta_{\max}$ .

We will provide a simple control law for the robots ensuring some form of non-instantaneous reciprocity. Our result in Section II will then allow us to establish (i) the convergence

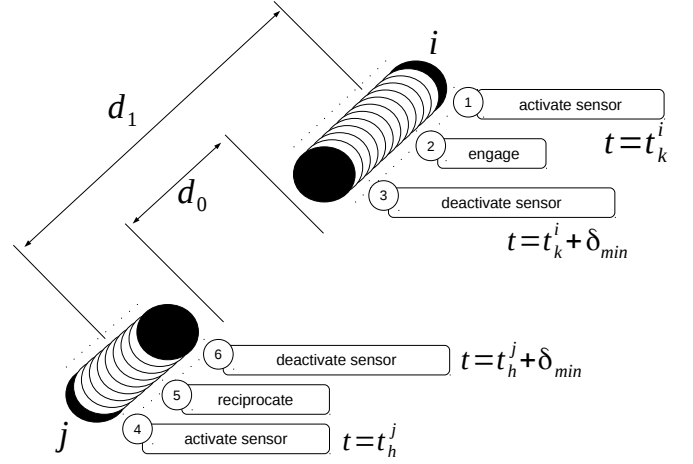


Fig. 3. Representations of the interactions taking place in the group of mobile robots with intermittent ultrasonic communication presented in Section IV. Events 1, 2 and 3 occur successively and so do events 4, 5 and 6. Event 4 occurs after event 1 and the following condition holds :  $t_h^j \in [t_k^i, t_k^i + \delta_{\max}]$ . When 2 occurs,  $a_{ij}(t) > 0$  and when 4 occurs  $a_{ji}(t) > 0$ . Proposition 4 provides conditions which guarantee that event 4 always takes place when event 2 has occurred, this ensures interaction reciprocity.

of all robot positions, and (ii) asymptotic practical consensus, that is, all robots eventually lie at a distance from each other smaller than a certain threshold. This threshold is proportional to  $d_0$ , the distance below which robots cannot sense each other. Since it converges, the system will not suffer from infinite oscillatory behaviors as in the example presented in Section III-B. To the best of our knowledge, such results cannot be obtained with any other convergence result available in the literature.

Our control law can be expressed as the following saturated consensus equation:

$$\dot{x}_i(t) = \text{sat} \sum_{j \in \mathcal{N}} b_{ij}(t) (x_j(t) - x_i(t)), \quad (6)$$

where the  $b_{ij}(t)$  will be specified later, and the function  $\text{sat} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is defined by

$$\text{sat}(x) = \begin{cases} \mu \cdot \frac{x}{\|x\|} & \text{if } \|x\| \geq \mu \\ x & \text{otherwise.} \end{cases}$$

The saturation guarantees that the velocity of each robot remains below its limit. We now explicit how the interaction weights  $b_{ij}$  are set. The idea is represented in Fig. 3: Agent  $i$  sets  $b_{ij}(t)$  to 1 when it starts monitoring the environment if its distance to  $j$  is larger than some appropriate radius  $d_1 > d_0$  (*engage*), or if its distance to  $j$  is larger than  $d_0$  and  $j$  has recently set  $b_{ji}(t)$  to 1 because it was at a distance larger than  $d_1$  from  $i$  at that time (*reciprocate*). The latter part of the algorithm is designed to ensure reciprocity, and the presence of  $d_1$  is needed to ensure that that  $i$  and  $j$  remain sufficiently distant for measurement to be made when  $i$  or  $j$  need to reciprocate.

Formally, we set  $b_{ij}(t) = 0$  by default, and set it to 1 in two cases:

$i$  engages

$$\exists k \in \mathbb{N}, t \in [t_k^i, t_k^i + \delta_{min}], \|x_i(t_k^i) - x_j(t_k^i)\| \geq d_1 \quad (7)$$

$i$  reciprocates

$$\begin{cases} \exists h \in \mathbb{N}, t \in [t_h^j, t_h^j + \delta_{min}], \|x_i(t) - x_j(t)\| \geq d_0 \text{ and} \\ \exists k \in \mathbb{N}, t_k^i \in [t_h^j - \delta_{max}, t_h^j], \|x_i(t_k^i) - x_j(t_k^i)\| \geq d_1. \end{cases} \quad (8)$$

*Remark 2:* Condition (7) can be easily implemented. To implement condition (8),  $i$  has to keep in memory the last activation time  $t_h^j$  at which the distance between  $i$  and  $j$  was higher than  $d_1$ . This could for example be achieved by having  $j$  sending a message to  $i$  at  $t_h^j$ .

Under these communication rules, we have the expected result :

*Proposition 4:* Consider system (6) where interaction occurs according to conditions (7) and (8). Also assume there holds

$$4\delta_{max} \cdot \mu \leq d_1 - d_0. \quad (9)$$

Then, the group of robots asymptotically achieves practical rendezvous:  $x_i^* = \lim_{t \rightarrow \infty} x_i(t)$  exists for every  $i \in \mathcal{N}$ , and

$$\lim_{t \rightarrow \infty} \Delta(t) \leq d_1,$$

where  $\Delta(t) = \max_{i,j \in \mathcal{N}} \|x_i(t) - x_j(t)\|$ .

*Proof:* Observe first that system (6) can be rewritten under the form of system (3) with

$$a_{ij}(t) = \frac{\mu \cdot b_{ij}(t)}{\left\| \sum_{k \in \mathcal{N}} b_{ik}(t)(x_k(t) - x_i(t)) \right\|} \quad (10)$$

if  $\left\| \sum_{k \in \mathcal{N}} b_{ik}(t)(x_k(t) - x_i(t)) \right\| \geq \mu$  and  $a_{ij}(t) = b_{ij}(t)$  otherwise. Since  $b_{ik}(t) = 0$  whenever  $\|x_k(t) - x_i(t)\| < d_0$ ,  $a_{ij}$  is upper bounded and thus is a non-negative measurable function, summable on bounded intervals of  $\mathbb{R}^+$ .

Moreover, since  $\Delta(t) = \max_{i,j \in \mathcal{N}} \|x_i(t) - x_j(t)\|$  is clearly nonincreasing, it follows from the definition of  $a_{ij}(t)$  that

$$a_{ij}(t) \geq b_{ij}(t) \min \left( \frac{\mu}{n\Delta(0)}, 1 \right), \quad (11)$$

where  $\Delta(0)$  is the initial group diameter.

In order to apply Theorem 2, we now show that the system under intermittent ultrasonic communication described above satisfies Assumption 3 with

$$\varepsilon = \min \left( \frac{\delta_{min}\mu}{n\Delta(0)}, \delta_{min} \right) \text{ and } T = 2\delta_{max}.$$

Let  $t \geq 0$  such that  $a_{ij}(t) > 0$ . Then,  $b_{ij}(t) > 0$  and at least one among conditions (7) and (8) is satisfied. Suppose first that condition (7) is satisfied and denote by  $k$  the integer such that  $t \in [t_k^i, t_k^i + \delta_{min}]$ . Clearly, condition (7) also holds for every  $s \in [t_k^i, t_k^i + \delta_{min}]$ .

We set  $\underline{t}_{ij} = t_k^i$  and  $\bar{t}_{ij} = t_k^i + 2\delta_{max} \geq t_k^i + \delta_{min}$ . Clearly, there holds  $t \in [\underline{t}_{ij}, \bar{t}_{ij}]$ , and  $\bar{t}_{ij} - \underline{t}_{ij} \leq 2\delta_{max} = T$ , so that

conditions (a) and (b) of Assumption 3 hold. Moreover, the non-negativity of  $a_{ij}$  implies that

$$\begin{aligned} \int_{\underline{t}_{ij}}^{\bar{t}_{ij}} a_{ij}(s) ds &\geq \int_{t_k^i}^{t_k^i + \delta_{min}} a_{ij}(s) ds \\ &\geq \min \left( \frac{\mu}{n\Delta(0)}, 1 \right) \int_{t_k^i}^{t_k^i + \delta_{min}} b_{ij}(s) ds \\ &= \min \left( \frac{\delta_{min}\mu}{n\Delta(0)}, \delta_{min} \right) = \varepsilon, \end{aligned}$$

where we have used (11) and the fact that  $b_{ij}(s) = 1$  for all  $s \in [t_k^i, t_k^i + \delta_{min}]$  since we have seen that condition (7) holds for those values. There remains to prove that  $\int_{\underline{t}_{ij}}^{\bar{t}_{ij}} a_{ji}(s) ds \geq \varepsilon$ .

Since  $t_{h+1}^j - t_h^j \leq \delta_{max}$  for all  $h \in \mathbb{N}$  and  $t_0^j \leq \delta_{max}$ , there exists  $h \in \mathbb{N}$  such that  $t_h^j \in [t_k^i, t_k^i + \delta_{max}]$ , and thus  $[t_h^j, t_h^j + \delta_{max}] \subseteq [t_k^i, t_k^i + 2\delta_{max}] = [\underline{t}_{ij}, \bar{t}_{ij}]$ . We show that the *reciprocate* condition (8) is satisfied for every  $s \in [t_h^j, t_h^j + \delta_{max}]$ . The second part of the condition directly follows from  $t_h^j \in [t_k^i, t_k^i + \delta_{max}]$ . For the first one, observe that  $\|\dot{x}_i\| \leq \mu$  (and the same holds for  $j$ ), and that  $\|x_i(t_k) - x_j(t_k)\| \geq d_1$  by assumption. Therefore, for any time  $s \in [t_h^j, t_h^j + \delta_{max}] \subseteq [t_k^i, t_k^i + 2\delta_{max}]$ , we have

$$\begin{aligned} \|x_i(s) - x_j(s)\| &\geq \|x_i(t_k) - x_j(t_k)\| - 4\mu\delta_{max} \\ &\geq d_1 - (d_1 - d_0) = d_0 \end{aligned}$$

for every  $s \in [t_h^j, t_h^j + \delta_{max}]$ , where we have used (9). As a consequence, the first part of condition (8) also holds, implying that  $b_{ij}(s) = 1$  for every  $s \in [t_h^j, t_h^j + \delta_{max}]$ . We get again

$$\int_{\underline{t}_{ij}}^{\bar{t}_{ij}} a_{ji}(s) ds \geq \min \left( \frac{\mu}{n\Delta(0)}, 1 \right) \int_{t_h^j}^{t_h^j + \delta_{min}} b_{ij}(s) ds = \varepsilon,$$

which achieves to show that Assumption 3 holds in that case.

Suppose now that  $a_{ij}(t) > 0$  because Condition (8) is satisfied at  $t$  for  $i, j$ . Then one can easily verify that condition 7 was satisfied for  $j, i$  for all  $s \in [t_j^k, t_j^k + \delta_{min}]$  for some  $t_j^k \in [t - \delta_{max}, t]$ , and an argument symmetric to that we have developed above shows that Assumption 3 also holds.

Since the weights  $a_{ij}(t)$  are upper-bounded, applying Theorem 2 (or more precisely its direct extension to  $\mathbb{R}^2$ , see Remark 1) shows that (i) the system converges:  $x_i^* = \lim_{t \rightarrow \infty} x_i(t)$  exists for every  $i$ , and (ii)  $x_i^* \neq x_j^*$  only if  $\int_0^\infty a_{ij}(t) dt < \infty$ .

To conclude the proof, suppose, to obtain a contradiction, that  $\lim_{t \rightarrow \infty} \Delta(t) > d_1$ , and thus that  $\|x_i^* - x_j^*\| > d_1$  for some  $i, j$ . The continuity of  $x$  implies that  $\|x_i(t) - x_j(t)\| > d_1$  for all  $t > s$  for some  $s$ , and in particular for all  $t_k^i > s$ . It follows then from the *engage* rule (7) that  $b_{ij}(t)$  would be set to 1 on infinitely many time intervals of length at least  $\delta_{min}$ . Besides, it follows from (11) that  $a_{ij}$  and  $b_{ij}$  remain within a bounded ratio, so that we would have  $\int_0^\infty a_{ij}(t) dt = \infty$ .

However, we have seen that  $x_i^* \neq x_j^*$  only if  $\int_0^\infty a_{ij}(t)dt < \infty$ , so there should hold  $x_i^* = x_j^*$ , in contradiction with our hypothesis. We have thus  $\lim_{t \rightarrow \infty} \Delta(t) \leq d_1$ . ■

Note that it is actually possible to have the robots converging to final positions within distances smaller than the  $d_1$  from Proposition 4 from each other. This can be achieved by decreasing their maximal speed  $\mu$  and the distance  $d_1$  when approaching convergence. Such more evolved control laws are however out of the scope of this section, where our goal was to demonstrate the use of our results from Section II.

## V. PROOFS

Before we prove Theorem 1, we provide several intermediate results. Our proof uses the following result on cut-balance discrete-time consensus systems.

*Theorem 5:* Let  $y : \mathbb{N} \rightarrow \mathbb{R}^n$  be a solution to

$$y_i(p+1) = \sum_{j=1}^n b_{ij}(p)y_j(p), \quad (12)$$

where  $b_{ij}(p) \geq 0$  and  $\sum_{j=1}^n b_{ij}(p) = 1$ . Suppose that the following assumptions hold:

a) *Lower bound on diagonal coefficients:* There exists a  $\beta > 0$  such that  $b_{ii}(p) \geq \beta$  for all  $i, p$ .

b) *Cut balance:* There exists a  $K' > 0$  such that for every  $p$  and non-empty proper subset  $S$  of  $\mathcal{N}$ , there holds

$$\sum_{i \in S, j \notin S} b_{ij}(p) \leq K' \sum_{i \in S, j \notin S} b_{ji}(p). \quad (13)$$

Then,  $y_i^* = \lim_{p \rightarrow \infty} y_i(p)$  exists for every  $i$ . Moreover, let  $G'(\mathcal{N}, E)$  be a directed graph where  $(j, i) \in E$  whenever  $\sum_{p=0}^\infty b_{ij}(p) = +\infty$ . There is a path from  $i$  to  $j$  in  $G$  if and only if there is a path from  $j$  to  $i$ , and in that case there holds  $y_i^* = y_j^*$ .

Theorem 5 is related to Theorem 1 in [25] for stochastic systems, and extends a part of Theorem 2 in [12], which also requires the existence of a uniform lower bound on the positive coefficients  $b_{ij}$ , that is, the existence of a  $\beta'$  such that  $b_{ij}(p) > 0 \Rightarrow b_{ij}(p) \geq \beta'$ . While this condition may appear minor, its absence is actually essential for our purpose, as it is in general not satisfied in the context of our proof. We provide in Section V-A a proof of Theorem 5 based on an idea used in [26]. We then complete the proof of Theorem 1 in Section V-B.

### A. Proof of Theorem 5

It follows directly from Lemma 1 in [13] that every weakly connected component of  $G'$  defined in Theorem 5 is strongly connected, that is, that there is a path from  $i$  to  $j$  if and only if there is a path from  $j$  to  $i$ .

We let  $B(p)$  be the (stochastic) matrix containing all coefficients  $b_{ij}(p)$ , that is,  $[B(p)]_{ij} = b_{ij}(p)$ . For  $q \geq p$ , we also let  $B[q : p] = B(q)B(q-1) \dots B(p+1)B(p)$ .

Observe that (12) can be rewritten as  $y(p+1) = B(p)y(p)$  or  $y(p) = B[p : 0]y(0)$ . We now call  $C_{G'}$  the set of matrices  $B \in \mathbb{R}^{n \times n}$  such that  $[B]_{ik} = [B]_{jk}$  for every  $k$  if  $(j, i) \in E'$ , where  $E'$  is the set of edges of the graph  $G'(\mathcal{N}, E')$  defined in the statement of Theorem 5. Observe that  $C_{G'}$  is a linear subspace of  $\mathbb{R}^{n \times n}$ . Moreover, since  $y(p) = B[p : 0]y(0)$ , the conclusion of Theorem 5 is equivalent to the convergence of  $B[p : 0]$  to a stochastic matrix  $B^* \in C_{G'}$ .

For given  $K' > 0$  and  $\beta' > 0$ , we now let  $P$  be the set of admissible matrices  $B$ , that this, the set of matrices  $B$  such that

- $B$  is stochastic:  $B\mathbf{1} = \mathbf{1}$  and  $[B]_{ij} \geq 0$ ,
- $[B]_{ii} \geq \beta$  for every  $i$ ,
- cut-balance:  $\sum_{i \in S, j \notin S} [B]_{ij} \leq K' \sum_{i \in S, j \notin S} [B]_{ji}$  for every subset  $S$  of  $\mathcal{N} = \{1, \dots, n\}$ .

The set  $P$  is a (finite) polytope with a finite number  $n_P$  of vertices  $V^{(1)}, V^{(2)}, \dots, V^{(n_P)}$ . We are going to represent our matrices  $B(p)$  as the expected value of random variables taking their values in the set of these vertices. Indeed, it

follows from its definition that every matrix  $B(p)$  belongs to  $P$ , and can thus be expressed as a convex combination of its vertices

$$B(t) = \sum_{\xi=1}^{n_P} \lambda^\xi(p) V^{(\xi)},$$

with  $\sum_{\xi=1}^{n_P} \lambda^\xi(p) = 1$  and  $\lambda^\xi(p) \geq 0$ . This decomposition is in general not unique, but we fix an arbitrary one. We then define for every  $p$  the (independent) random variables  $R(p) \in \mathbb{R}^{n \times n}$  taking their values in the set of vertices of  $P$ , with  $R(p) = V^{(\xi)}$  with probability  $\lambda(p)^{(\xi)}$ . It follows directly that  $\mathbb{E}(R(p)) = B(p)$ . Moreover, since the  $R(p)$  are independent, there holds

$$\begin{aligned} \mathbb{E}(R[p : 0]) &= \mathbb{E}(R(p)R(p-1) \dots R(0)) \\ &= \mathbb{E}(R(p))\mathbb{E}(R(p-1)) \dots \mathbb{E}(R(0)) \\ &= B(p)B(p-1) \dots B(0) = B[p : 0]. \end{aligned}$$

To analyze the convergence of  $B[p : 0]$ , we analyze thus the convergence of the  $R[p : 0]$ .

Let us fix a realization of the random sequence of matrices  $R(p)$  and consider the corresponding product  $R[p : 0]$ . For every time  $q$ ,  $R(q)$  belongs by definition to  $P$  and satisfies thus condition (a) and (b) of the Theorem. Moreover, since the  $R(q)$  is always a vertex of  $P$ , it can take only finitely many values. One can thus find a  $\beta' > 0$  such that  $[R(q)]_{ij}(q) > 0 \Rightarrow [R(q)]_{ij}(q) \geq \beta'$  (it suffices to take the smallest nonzero entry over all  $V^{(\xi)}$ ). We can therefore apply Theorem 2 in [12], which implies that  $R^* = \lim_{p \rightarrow \infty} R[p : 0]$  exists, and that if  $\sum_{p=0}^\infty [R(p)]_{ij} = \infty$ , then  $[R^*]_{ik} = [R^*]_{jk}$  for every  $k$ .

We now analyze the condition  $\sum_{p=0}^\infty [R(p)]_{ij} = \infty$ . Since  $[R(p)]_{ij} \leq 1$ , we have  $\mathbb{E}([R(p)]_{ij}) \leq \mathbb{P}([R(p)]_{ij} > 0)$ . Thus, since the  $R(p)$  are independent, it follows from the second Borel-Cantelli Lemma that if  $\sum_{p=0}^\infty \mathbb{E}([R(p)]_{ij}) = \infty$  then  $[R(p)]_{ij} > 0$  occurs infinitely often with probability 1.

Since nonzero entry in  $[R(p)]_{ij}$  are lower bounded by  $\beta'$ , we have that  $\sum_{p=0}^{\infty} [R(p)]_{ij} = \infty$  with probability 1 if  $\sum_{p=0}^{\infty} \mathbb{E}([R(p)]_{ij}) = \infty$  that is if  $\sum_{p=0}^{\infty} [B(p)]_{ij} = \infty$ , and thus if  $(j, i) \in E'$ . As a consequence  $R^* \in C_{G'}$  with probability 1.

The conclusion of Theorem 5 follows then from the application of the following Proposition to the sequence of random variables  $R[p : 0] = R(p)R(p-1) \dots R(0)$ , the set  $S$  of stochastic matrices, and the linear subspace  $C_{G'}$ . Indeed, since  $R[p : 0]$  always converges, and its limit belongs to  $C_{G'}$  with probability 1, it implies that  $B[p : 0] = \mathbb{E}(R[p : 0])$  converges to some matrix  $B^*$  in  $C_{G'}$ . Moreover, since every  $B[p, 0]$  is stochastic and the set of stochastic matrices is compact,  $B^*$  is also stochastic.

*Proposition 6:* Let  $\{Z_p : p \in N\}$  be a (discrete-time) stochastic process, where  $Z_p$  takes its value in a bounded subset  $S \subset \mathbb{R}^n$  for every  $p$ .

- (a) If  $\lim_{p \rightarrow \infty} z_p$  exists for every realization of  $Z$ , then  $\lim_{t \rightarrow \infty} \mathbb{E}(z_p)$  exists.
- (b) Let  $W$  be a linear subspace of  $\mathbb{R}^n$ . If  $\lim_{t \rightarrow \infty} z_p$  exists and belongs to  $W$  almost surely, then  $\lim_{p \rightarrow \infty} \mathbb{E}(z_p)$  exists and belongs to  $W$ .

We remark that the assumptions of Proposition 6 do not require (almost) all realizations of  $Z$  to converge to a same value, but only that (almost) every realization of  $Z$  converges.

*Proof:* To prove (a), observe that our stochastic process defines a measure space  $(\Omega, A, \mu)$ , where  $\Omega$  is the sample space (set of all possible realizations of  $Z$ ),  $A$  the functions defined on  $\Omega$  and  $\mu$  the measure defined by the probabilities. The random variables  $Z_p$  form a sequence of real-valued functions on  $\Omega$ .

Since  $\lim_{p \rightarrow \infty} z_p$  exists for every realization, this sequence of functions is pointwise convergent, and we call  $Z^*$  its limit. Moreover, the sequence of functions  $Z_p$  is uniformly bounded because they all take their values in the bounded set  $S$ . We can therefore apply the bounded convergence theorem, which yields

$$\lim_{p \rightarrow \infty} \mathbb{E}(Z_p) = \lim_{p \rightarrow \infty} \int_{\Omega} Z_p d\mu = \int_{\Omega} Z^* d\mu,$$

and thus the existence of  $\lim_{p \rightarrow \infty} \mathbb{E}(Z_p)$ .

Let us now prove (b). Since  $\lim_{p \rightarrow \infty} z_p \in W$  almost surely, the distance  $d(z_p, W)$  between  $z_p$  and  $W$  converges to zero almost surely. Convergence with a probability 1 implies convergence in distribution (see for example Theorem 25.2 in [1]), and therefore the convergence of the expected values, so that  $\lim_{t \rightarrow \infty} \mathbb{E}(d(z_p, W)) = 0$ . Since  $W$  is a linear and hence convex set,  $d(\cdot, W)$  is a convex function. It follows then from Jensen's inequality that  $d(\mathbb{E}(z_p), W) \leq \mathbb{E}(d(z_p, W))$ , and thus that  $d(\mathbb{E}(z_p), W)$  also converges to 0. The limit  $\lim_{p \rightarrow \infty} \mathbb{E}(z_p)$  must thus be a distance 0 from  $W$ , that is in  $W$ . ■

## B. Proof of Theorem 1

To apply Theorem 5, we focus on the values taken by the states at times  $t_p$ . Remember that the sequence of times  $t_p$

defines the intervals over which the integral reciprocity is satisfied.

*Lemma 7:* The sequence of states  $(x(t_p))$  can be written as the trajectory of the discrete-time consensus system obtained by sampling (3)

$$x_i(t_{p+1}) = \sum_{j \in \mathcal{N}} \phi_{ij}(p) \cdot x_j(t_p), \quad (14)$$

where the weights  $\phi_{ij}(p)$  are non-negative and satisfy  $\sum_{j \in \mathcal{N}} \phi_{ij}(p) = 1$ . This sampled system always exists and is unique for given weights  $a_{ij}(t)$  and sampling times  $t_p$ , and the weights  $\phi_{ij}(p)$  are independent of states  $x(t)$ .

In particular, if  $x_j(t_p) = 1$  for  $j \in S$  and  $x_k(t_p) = 0$  for  $k \notin S$ , for some  $S \subseteq \mathcal{N}$ , there holds

$$\sum_{j \in S} \phi_{ij}(p) = x_i(t_{p+1}). \quad (15)$$

*Remark 3:* The equality ((15)) provides a way of computing or bounding certain sums of the weights  $\phi_{ij}(p)$  by considering the evolution of the systems starting from 'artificial' states, where  $x_j(t_p) = 1$  for some agents and  $x_k(t_p) = 0$  for the others.

Note that these artificial states are only a formal tool to compute weights  $\phi_{ij}(p)$ , and their use does not result in any loss of generality.

*Proof:* Denote  $\Phi(t, T)$  the fundamental matrix of the linear dynamics (3) which is uniquely defined [10] such that

$$x(T) = \Phi(t, T)x(t).$$

We define  $\phi_{ij}(p)$  as the  $ij$ -th coefficient of matrix  $\Phi(t_p, t_{p+1})$ . So, the  $\phi_{ij}(p)$  are unique and equation (14) is satisfied. Moreover, for given weights  $a_{ij}(t)$ , matrix  $\Phi(t, T)$  is independent of state  $x(t)$  and so are the weights  $\phi_{ij}(p)$ . So if we assume artificial states  $x_j(t_p) = 1$  for  $j \in S$  and  $x_k(t_p) = 0$  for  $k \notin S$ , we obtain (15) from equation (14). Since system (3) preserves the nonnegativity of the states It follows from equation (15) applied to  $S = \{j\}$  that  $\phi_{ij}(p) \geq 0$  for every  $i, j, p$ .

Finally, we can use the Peano-Baker to show that  $\sum_{j \in \mathcal{N}} \phi_{ij}(p) = 1$  : the formula gives  $\Phi(t, T)$  as the limit of a recursive series

$$\Phi(t, T) = \lim_{n \rightarrow \infty} M_n(T)$$

with

$$M_0(\tau) = I \text{ and } M_{n+1}(\tau) = I - \int_t^\tau L(s)M_n(s)ds,$$

where  $I$  is the identity matrix and  $L(s)$  the Laplacian matrix of  $A(s) = (a_{ij}(s))$ , i.e. with diagonal elements equal to  $\sum_{j \in \mathcal{N}} a_{ij}(s)$  and off-diagonal elements equal to  $-a_{ij}(s)$ . Since  $L \cdot \mathbf{1} = 0$  with  $\mathbf{1}$  the vector of all ones, we have from the recursive equation that  $M_n \cdot \mathbf{1} = \mathbf{1}$  and by continuity,  $\Phi(t, T) \cdot \mathbf{1} = \mathbf{1}$ , thus  $\sum_{j \in \mathcal{N}} \phi_{ij}(p) = 1$ . ■

To obtain more insights on the discrete-time weights  $\phi_{ij}$ , we give the next proposition which bounds the discrete-time weights  $\phi_{ij}$  using the continuous-time weights  $a_{ij}$ .



*Proposition 8:* Under the uniform bound Assumption 2, we have for all proper subset of agents  $S$ , for all  $p \geq 0$ ,

$$G \cdot \sum_{i \in S} \int_{t_p}^{t_{p+1}} a_{ij}(t) dt \leq \sum_{i \in S} \phi_{ij}(p) \leq n \cdot \sum_{i \in S} \int_{t_p}^{t_{p+1}} a_{ij}(t) dt,$$

with  $G = \exp(-2nM)/n$ .

*Proof:* Let  $p \in \mathbb{N}$  and  $S$  a proper subset of  $\mathcal{N}$ . We assume that

$$\forall i \in S, x_i(t_p) = 0 \text{ and } \forall j \in S, x_j(t_p) = 1, \quad (16)$$

as suggested in Remark 3.

We first show the left inequality. We show that starting from state (16) at time  $t_p$  no agent  $j \notin S$  can be arbitrarily close to 0 at time  $t_{p+1}$ . We have for all  $\tau \in [t_p, t_{p+1}]$ ,

$$\begin{aligned} x_j(\tau) &= x_j(t_p) + \int_{t_p}^{\tau} \sum_{k \in \mathcal{N}} a_{jk}(t) \cdot (x_k(t) - x_j(t)) dt \\ &\geq x_j(t_p) - \int_{t_p}^{\tau} \sum_{k \in \mathcal{N}} a_{jk}(t) \cdot x_j(t) dt, \end{aligned}$$

where we used  $x_k(t) \geq 0, k \in \mathcal{N}$ . We use Gronwall's inequality and Assumption 2 (upper bound on interactions on each  $[t_p, t_{p+1}]$ ) to obtain

$$j \notin S \Rightarrow x_j(\tau) \geq e^{-nM}, \forall \tau \in [t_p, t_{p+1}] \quad (17)$$

We rely on bound (17) to prove that, due to attraction from agents not in  $S$ , all states  $x_i(t_{p+1})$ , for  $i \in S$ , cannot be arbitrarily close to 0 at time  $t_{p+1}$ .

Let now  $h \in S$  be such that

$$\sum_{j \notin S} \int_{t_p}^{t_{p+1}} a_{hj} = \max_{i \in S} \sum_{j \notin S} \int_{t_p}^{t_{p+1}} a_{ij},$$

i.e., agent  $h$  is the element in  $S$  receiving the highest influence from the rest of the group. In particular, there holds

$$\sum_{j \notin S} \int_{t_p}^{t_{p+1}} a_{hj} \geq \frac{1}{n} \sum_{i \in S} \sum_{j \notin S} \int_{t_p}^{t_{p+1}} a_{ij} \quad (18)$$

Using that  $x_i \geq 0$  for  $i \in S$  and the lower bound (17) on  $x_j$  for  $j \notin S$ , we have for all  $\tau \in [t_p, t_{p+1}]$ ,

$$\begin{aligned} x_h(\tau) &= x_h(t_p) + \int_{t_p}^{\tau} \sum_{j \notin S} a_{hj}(x_j - x_h) + \int_{t_p}^{\tau} \sum_{k \in \mathcal{N}} a_{hk}(x_k - x_h) \\ &\geq x_h(t_p) + \int_{t_p}^{\tau} \sum_{j \notin S} a_{hj} x_j - \int_{t_p}^{\tau} \sum_{k \in \mathcal{N}} a_{hk} x_h \\ &\geq e^{-nM} \int_{t_p}^{\tau} \sum_{j \notin S} a_{hj} - \int_{t_p}^{\tau} \sum_{k \in \mathcal{N}} a_{hk} x_h, \end{aligned}$$

where we have also used  $x_h(t_p) = 0$ . It follows then from Gronwall's inequality.

$$x_h(t_{p+1}) \geq e^{-nM} \int_{t_p}^{t_{p+1}} e^{-\int_{t_p}^{\tau} \sum_{k \in \mathcal{N}} a_{hk}} \sum_{j \notin S} a_{hj}. \quad (19)$$

We now bound the term in the exponential using Assumption 2 (upper bound) together with (18) to write

$$x_h(t_{p+1}) \geq \frac{1}{n} e^{-2nM} \sum_{i \in S, j \notin S} \int_{t_p}^{t_{p+1}} a_{ij}. \quad (20)$$

Moreover,  $\phi_{ij}(p) \geq 0$  and equation (15) yield

$$\sum_{i \in S} \sum_{j \notin S} \phi_{ij}(p) \geq \sum_{j \notin S} \phi_{hj}(p) = x_h(t_{p+1}).$$

We conclude the first part of the proof combining the two previous equations. We now turn to the second inequality.

Denote, for  $t \in [t_p, t_{p+1}]$ ,

$$\bar{x}_S(t) = \max_{i \in S} x_i(t) = \bar{x}_S(t_p) + \int_{t_p}^t \sum_{k \in \mathcal{N}} a_{m(\tau)k} (x_k - \bar{x}_S) d\tau,$$

where  $m(\tau) \in S$  is chosen such that  $x_{m(\tau)}(\tau) = \bar{x}_S(\tau)$ . The last equality has been shown in [12, Proposition 2]. Notice that the choice of state (16) implies  $\bar{x}_S(t_p) = 0$ . Since  $x_j \leq 1$  for  $j \notin S$  and  $x_i \leq \bar{x}_S \leq 1$  for  $i \in S$ ,

$$\begin{aligned} \bar{x}_S(t) &\leq \int_{t_p}^t \sum_{j \notin S} a_{m(\tau)j} (1 - \bar{x}_S) d\tau \\ &\leq \int_{t_p}^t \sum_{i \in S, j \notin S} a_{ij} (1 - \bar{x}_S). \end{aligned}$$

The Gronwall's inequality yields

$$\bar{x}_S(t_{p+1}) \leq 1 - e^{-\int_{t_p}^{t_{p+1}} \sum_{i \in S, j \notin S} a_{ij}} \leq \int_{t_p}^{t_{p+1}} \sum_{i \in S, j \notin S} a_{ij}. \quad (21)$$

We conclude with

$$\sum_{i \in S, j \notin S} \phi_{ij}(p) = \sum_{i \in S} x_i(t_{p+1}) \leq n \bar{x}_S(t_{p+1}).$$

■

The previous proposition serves to transpose the cut-balance assumption provided in Theorem 1 to the discrete-time weights  $\phi_{ij}(p)$ . In particular, we can now show that the condition of Theorem 5 are satisfied, more precisely, we have Lemma 9 regarding weights  $\phi_{ij}(p)$ .

*Lemma 9:* The following properties hold :

- There exists a lower bound  $\beta > 0$  on diagonal elements:  $\phi_{ii}(p) \geq \beta$ , for all  $p$  and  $i$ .
- The weights  $\phi_{ij}(p)$  satisfy the cut balance assumption (13) for some  $K'$  determined by the constants  $K$  and  $M$  of Assumptions 1 and 2.

Note that (b) would in general not be true for certain stronger forms or reciprocity. In particular,  $\int_{t_p}^{t_{p+1}} a_{ij}(t) dt \leq K \int_{t_p}^{t_{p+1}} a_{ji}(t) dt$  does not imply the existence of a  $K'$  such that  $\phi_{ij}(p) \leq K' \phi_{ji}(p)$ .

*Proof:* The proof of (a) is as follows. For arbitrary  $k \in \mathcal{N}$  and  $p$ , we suppose that  $x_k(t_p) = 0$ , and  $x_i(t_p) = 1$  for every  $i \neq k$ . A reasoning similar to that leading to (17) in

the proof of Proposition 8 shows that  $x_k(t_{p+1}) \leq 1 - e^{-nM}$ . It follows then from Lemma 7 applied to  $S = \{k\}$  that  $\sum_{j \in \mathcal{N}, j \neq k} \phi_{kj}(p) \leq 1 - e^{-nM}$ , and thus that  $\phi_{kk}(p) \geq e^{-nM}$ , which establishes (a).

We now prove statement (b). Proposition 8 applied to  $S$  states that

$$\sum_{i \in S} \phi_{ij}(p) \leq n \cdot \sum_{i \notin S} \int_{t_p}^{t_{p+1}} a_{ij}(t) dt. \quad (22)$$

On the other hand, applying the second inequality of the same proposition, applied to  $\mathcal{N} \setminus S$  yields

$$G \cdot \sum_{i \notin S} \int_{t_p}^{t_{p+1}} a_{ij}(t) dt \leq \sum_{i \notin S} \phi_{ij}(p),$$

which can be rewritten as

$$G \cdot \sum_{i \in S} \int_{t_p}^{t_{p+1}} a_{ji}(t) dt \leq \sum_{i \in S} \phi_{ji}(p). \quad (23)$$

Statement (b) with  $K' = n/G$  follows then directly from Assumption 1 and the inequalities (22) and (23). ■

*Proof:* [of Theorem 1]

Since Lemma 9 is satisfied, Theorem 5 applies. Thus, the sequence  $x(t_p)$  converges to some  $x^* \in \mathbb{R}^n$ . Denote  $G' = (\mathcal{N}, E)$  the directed graph where  $(j, i) \in E$  whenever  $\sum_{p=0}^{\infty} \varphi_{ij}(p) = +\infty$ . Theorem 5 also implies that  $x_i^* = x_j^*$  if  $i$  and  $j$  belong to the same connected components of the graph  $G'$ . This graph only has strong components which are fully disconnected to each other and in which consensus takes place for the discrete-time system  $y(p) = x(t_p)$ . The graph  $G$  of persistent interactions defined in the statement of Theorem 1 is in general different from  $G'$ . However, as a direct corollary of Proposition 8, we have that  $G$  and  $G'$  have the same connected components.

It remains to show that the continuous-time function  $x(t)$  converges to the same  $x^*$  as sequence  $(x(t_p))$ . We prove this by showing that for each  $S \subseteq \mathcal{N}$  strongly connected component of  $G$  (or of  $G'$ ), both the minimum  $\underline{x}_S(t) = \min_{i \in S} x_i(t)$  and the maximum  $\bar{x}_S(t) = \max_{i \in S} x_i(t)$  converge to the same value. Since  $S$  is a connected component of  $G$ , the integral influence  $\sum_{i \in S, j \notin S} \int_0^{\infty} a_{ij}$  is finite. For any  $\mu < 0$ , there exists some  $T_\mu \geq 0$  such that

$$\sum_{i \in S, j \notin S} \int_{T_\mu}^{\infty} a_{ij} < \mu.$$

For all  $v > u \geq T_\mu$ , we have, using notation  $m(\tau) \in S$

chosen such that  $x_{m(\tau)}(\tau) = \bar{x}_S(\tau)$  as before,

$$\begin{aligned} \bar{x}_S(v) - \bar{x}_S(u) &\leq \sum_{j \notin S} \int_u^v a_{m(\tau)j} (x_j(\tau) - x_{m(\tau)}(\tau)) d\tau \\ &\leq \sum_{j \notin S} \int_u^v a_{m(\tau)j} |x_j(\tau) - x_{m(\tau)}(\tau)| d\tau \\ &\leq \sum_{i \in S, j \notin S} \int_u^v a_{m(\tau)j} |x_j(\tau) - x_i(\tau)| d\tau \\ &\leq \sum_{i \in S, j \notin S} \int_{T_\mu}^{\infty} a_{ij} |x_j - x_i| \\ &\leq \mu \Delta(0), \end{aligned}$$

where  $\Delta(0) = \max_{i \in \mathcal{N}} x_i(0) - \min_{i \in \mathcal{N}} x_i(0)$ . This shows that  $\bar{x}_S$  converges in the sense of Cauchy, thus it converges. Since sub-sequence  $(\bar{x}_S(t_p))$  converges to  $x_i^*$  for some  $i \in S$ , it must be that  $\lim_{t \rightarrow +\infty} \bar{x}_S(t) = x_i^*$ . We can apply the same reasoning to show that  $\underline{x}_S$  also converges  $\lim_{p \rightarrow +\infty} \underline{x}_S(t_p) = x_i^*$ . We conclude that for all  $i \in S$ ,  $x_i(t)$  converge to the same limit  $x_i^*$ . ■

## VI. PROOF OF THEOREM 2

For concision, we say that an unordered pair  $\{i, j\} = \{j, i\}$  is *active* over an interval  $I$  if  $\int_{t \in I} a_{ij}(t) dt \geq \varepsilon$  and  $\int_{t \in I} a_{ji}(t) dt \geq \varepsilon$ . The following Lemma compiles some properties following from that definition.

*Lemma 10:*

- Consider two intervals  $I, J$  with  $I \subseteq J$ . If  $\{i, j\}$  is active over  $I$ , it is active over  $J$ .
- Consider two intervals  $I, J$  with  $I \subseteq J$ . If  $\{i, j\}$  is not active over  $J$ , it is not active over  $I$ .
- Under Assumption 3, if  $a_{ij}(t) > 0$ , then  $\{i, j\}$  is active over  $[t - T, t + T]$ .
- Under Assumption 3, if  $\{i, j\}$  is not active over  $[t, t']$ , then  $a_{ij}(s) = a_{ji}(s) = 0$  for all  $s \in [t + T, t' - T]$ .

The next Proposition is the core of our proof, it allows building a sequence of times  $t_k$  valid for Assumptions 1 and 2.

*Proposition 11:* Suppose that Assumption 3 is satisfied, and let  $M = M_1 + M_2$  where  $M_1, M_2$  are any constant satisfying

$$M_2 > n(n-1)T + T \text{ and } M_1 \geq M_2 + T. \quad (24)$$

Then, there exists a sequence  $t_0, t_1, \dots$  with  $t_0 = 0$ , and  $t_{k+1} - t_k \leq M$ , such that the following condition  $A_k$  holds for every  $k$ .

$$A_k : \forall i, j \in \mathcal{N} \text{ distinct, } A1_k \text{ or } A2_k,$$

with

$$\begin{cases} A1_k : \forall t \in [t_k, t_{k+1}], a_{ij}(t) = 0, \\ A2_k : \{i, j\} \text{ is active over } [t_k, t_{k+1}]. \end{cases}$$

The proof of Proposition 11 is based on an induction that makes use of the intermediate condition  $B_k$  :

$$B_k : \forall i, j \in \mathcal{N} \text{ distinct, } B1_k \text{ or } B2_k,$$

with

$$\begin{cases} B1_k : \forall t \in [t_k, t_k + T], a_{ij}(t) = 0, \\ B2_k : \{i, j\} \text{ is active over } [t_k, t_k + M_1]. \end{cases}$$

The next Lemma concerns the initialization of the induction.

*Lemma 12:* Suppose that  $a_{ij}(t) = 0$  for all  $t \leq 0$ , and let  $t_0 = 0$ . Then Condition  $B_0$  holds.

*Proof:* Suppose that  $B1_0$  does not hold, i.e.  $a_{ij}(t) > 0$  for some  $t \in [0, T]$ . Then, Lemma 10(c) gives that  $\{i, j\}$  is active over  $[t - T, t + T]$  and thus from Lemma 10(a) that  $\{i, j\}$  is active over  $[\min(0, t - T), 2T]$ . Since  $a_{ij}(t') = 0$  for all  $t' < 0$ , it follows then that  $\{i, j\}$  is active over  $[0, 2T]$  and since according to equation (24),  $M_1 \geq M_2 + T > n(n - 1) + 2T \geq 2T$ ,  $\{i, j\}$  is active over  $[0, M_1]$  (again thanks to Lemma 10(a)). Thus  $B2_0$  holds and so does  $B_0$ . ■

*Proposition 13 (Inductive case):* If there exists  $t_k$  such that condition  $B_k$  holds, then there exists  $t_{k+1} \leq t_k + M_1 + M_2$  for which conditions  $A_k$  and  $B_{k+1}$  hold.

*Proof:*

Let us introduce the two following sets of unordered pairs of agents for every  $t \in [t_k, t_k + M_1 + M_2]$ .

- $R_t \subseteq \{\{i, j\} | i, j \in \mathcal{N}, i \neq j\}$ : set of pairs  $\{i, j\}$  which are active over time interval  $[t_k, t]$ .
- $V_t \subseteq \{\{i, j\} | i, j \in \mathcal{N}, i \neq j\}$ : set of pairs  $\{i, j\}$  for which  $a_{ij}(t') = a_{ji}(t') = 0$  for all  $t' \in [t, t_k + M_1 + M_2]$  (i.e. set of pairs where there is no interaction between  $t$  and  $t_k + M_1 + M_2$ ).

Note that for all  $t_k \leq t \leq s \leq t_k + M_1 + M_2$ , there holds  $R_t \subseteq R_s$  and  $V_t \subseteq V_s$ , so that these sets are non-decreasing with time. The non-decrease of  $V_s$  is trivial while that of  $R_s$  follows directly from Lemma 10(a).

We now build a  $t_{k+1}$  using the Algorithm 1, which we prove to always successfully terminate. We first prove that Claims 1 and 2 hold, and then show how this implies the statement of this Proposition.

---

**Algorithm 1** Selection of  $t_{k+1}$

---

**Require:**  $t_k$  satisfies  $B_k$

Set  $\bar{t} = t_k + M_1$

**Switch over cases 0 to 3 :**

**Case 0:**  $\bar{t} \geq t_k + M_1 + M_2 - T$ : STOP, FAILURE

**Case 1:** conditions  $A_k$  and  $B_{k+1}$  are satisfied taking  $t_{k+1} = \bar{t}$ . STOP, SUCCESS.

**Case 2:** Condition  $A_k$  does not hold taking  $t_{k+1} = \bar{t}$ .

*Claim 1:* There exists  $\{i, j\} \notin R_{\bar{t}}$  belonging to  $R_{\bar{t}+T}$ .

**Then** set  $\bar{t} = \bar{t} + T$  and iterate.

**Case 3:** Condition  $B_{k+1}$  does not hold taking  $t_{k+1} = \bar{t}$ .

*Claim 2:* There exists  $\{i, j\} \notin V_{\bar{t}}$  belonging to  $V_{\bar{t}+T}$ .

**Then** set  $\bar{t} = \bar{t} + T$  and iterate.

---

*Claim 1:*

In Case 2 of Algorithm 1, condition  $A_k$  does not hold. There exists thus  $\{i, j\}$  such that  $A1_k$  does not hold, i.e.  $a_{ij}(t) > 0$  for some  $t \in [t_k, \bar{t}]$ , and  $A2_k$  does not hold, i.e.  $\{i, j\}$  is not active over  $[t_k, \bar{t}]$ .

The fact that  $A2_k$  does not hold implies by definition of  $R_{\bar{t}}$  that  $\{i, j\} \notin R_{\bar{t}}$ . Let us now show that  $t \in [\bar{t} - T, \bar{t}]$ . The fact that  $A2_k$  does not hold together with Lemma 10(d) implies that  $a_{ij}(t') = 0$  for all  $t' \in [t_k + T, \bar{t} - T]$ . So either  $t \in [t_k, t_k + T]$  or  $t \in [\bar{t} - T, \bar{t}]$ . We show that the first case is impossible: Since  $\{i, j\}$  is not active over  $[t_k, \bar{t}]$ , and  $\bar{t} \geq t_k + M_1$ , Lemma 10(b) implies that  $\{i, j\}$  is not active over  $[t_k, t_k + M_1]$ , and thus that,  $B2_k$  does not hold. However, we know by hypothesis that  $B_k$  holds. Thus,  $B1_k$  holds :  $t \notin [t_k, t_k + T]$ , and as a consequence,  $t \in [\bar{t} - T, \bar{t}]$ .

It follows then from Lemma 10(c) that  $\{i, j\}$  is active over  $[t - T, t + T]$  and from Lemma 10(a) that it is active over  $[\bar{t} - 2T, \bar{t} + T]$ . Since  $\bar{t} \geq t_k + M_1 > t_k + 2T$ , the pair  $\{i, j\}$  is active over  $[t_k, \bar{t} + T] : \{i, j\} \in R_{\bar{t}+T}$ , which achieves proving claim 1.

*Claim 2:*

Since condition  $B_{k+1}$  does not hold, there is a pair  $\{i, j\}$  that satisfies neither  $B1_{k+1}$  nor  $B2_{k+1}$ , that is, one for which  $a_{ij}(t) > 0$  for some  $t \in [\bar{t}, \bar{t} + T]$ , and for which  $\{i, j\}$  is not active over  $[\bar{t}, \bar{t} + M_1]$ . Since  $\bar{t} \leq t_k + M_1 + M_2 - T$  for otherwise we would have been in case 0, the  $t \in [\bar{t}, \bar{t} + T]$  for which  $a_{ij}(t) > 0$  lies in  $[\bar{t}, t_k + M_1 + M_2]$ , which implies that  $\{i, j\} \notin V_{\bar{t}}$  by definition of  $V_{\bar{t}}$ . We now show that it belongs to  $V_{\bar{t}+T}$ .

By Lemma 10(d), since  $\{i, j\}$  is not active over  $[\bar{t}, \bar{t} + M_1]$ ,  $a_{ij}(t') = a_{ji}(t') = 0$  for all  $t' \in [\bar{t} + T, \bar{t} + M_1 - T]$ . Also, by definition,  $\bar{t} \geq t_k + M_1$  and  $M_1 \geq M_2 + T$ , so that

$$\bar{t} + M_1 - T \geq t_k + M_1 + M_1 - T \geq t_k + M_1 + M_2.$$

Thus,  $a_{ij}(t') = a_{ji}(t') = 0$  for all  $t' \in [\bar{t} + T, t_k + M_1 + M_2]$  and  $\{i, j\} \in V_{\bar{t}+T}$ .

To complete the proof of Proposition 13, we show that Algorithm 1 stops and that when it does, the choice  $t_{k+1} = \bar{t}$  satisfies conditions  $A_k$  and  $B_{k+1}$ . If case 2 or 3 applies,  $\bar{t}$  increases by  $T$ . Otherwise the algorithm stops. In case 2, it follows from Claim 1 that the size of  $R_{\bar{t}}$  increases by at least 1, and in case 3, it follows from Claim 2 that the size of  $V_{\bar{t}}$  increases by at least 1. Since both  $R_{\bar{t}}$  and  $V_{\bar{t}}$  are sets of unordered pairs of distinct nodes, their size cannot exceed  $n(n-1)/2$ . Therefore, Cases 2 and 3 do not apply more than  $n(n-1)/2$  times each. In particular, case 0 or 1 must apply for some  $\bar{t} \leq t_k + M_1 + n(n-1)T$  (remembering that  $\bar{t}$  is initially  $t_k + M_1$ ), at which stage the algorithm stops. Now since according to equation (24),  $M_2 > n(n-1)T + T$ , case 0 or 1 apply for  $\bar{t} < t_k + M_1 + M_2 - T$ , so that case 1 must apply first, and the algorithm produces thus a  $t_{k+1}$  satisfying  $t_{k+1} - t_k \leq M_1 + M_2$  for which  $A_k$  and  $B_{k+1}$  are satisfied. ■

The proof of Proposition 11 is then a direct consequence of Lemma 12 and Proposition 13.

*Proof:* [of Theorem 2] We show that the sequence  $t_k$  built in Proposition 11 is valid for Assumption 1 and 2. Observe first that since the  $a_{ij}(t)$  are assumed to be uniformly bounded and since  $t_{k+1} - t_k \leq M$ , there clearly

holds  $\int_{t_k}^{t_{k+1}} a_{ij}(t)dt < M'$  for some  $M'$  and all  $i, j$  and  $t_k$ , so that Assumption 2 holds. Moreover, it follows from Proposition 11 that either  $a_{ij}(t) = a_{ji}(t) = 0$  for all  $t \in [t_k, t_{k+1}]$ , or  $\int_{t_k}^{t_{k+1}} a_{ij}(t)dt \geq \varepsilon$  and  $\int_{t_k}^{t_{k+1}} a_{ji}(t)dt \geq \varepsilon$ . Since the latter integrals are also bounded by  $M'$ , there holds

$$\int_{t_k}^{t_{k+1}} a_{ij}(t)dt \leq \frac{M'}{\varepsilon} \int_{t_k}^{t_{k+1}} a_{ji}(t)dt,$$

which implies that Assumption 1 also holds. ■

## VII. CONCLUSION

In this paper, we have developed a convergence result for continuous-time consensus systems. This result is based on a new assumption which allows for non-instantaneous reciprocity. Unlike previous studies, we only assume that reciprocity takes place on average over contiguous time intervals. This assumption is appropriate for various classes of systems (including classes of broadcasting, gossiping, and self-triggered system where communication is not necessarily synchronous). We have shown that the integral reciprocity is not alone sufficient for convergence. Oscillatory behaviors may take place. Thus, we provided a companion assumption which assumes that, over intervals where reciprocity occurs, the amount of interaction is always uniformly bounded. Under these two assumptions, we have proven that the trajectory to the consensus system always converges. Moreover, consensus takes place among agents in clusters of the graph of persistent interactions. We have also particularized our result to a class of systems satisfying a local pairwise form of reciprocity.

Apart from the integral reciprocity and uniform bound, our result does not make any assumption on the interactions between agents, and allows in particular for arbitrary long periods during which the system is idle. As a consequence, it is in general impossible to give absolute bounds on the speed of convergence under the assumptions that we have made. However, future works could relate the speed of convergence to the amount of interactions having taken place in the system, as in [18].

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